BOSWELL-BÈTA

James Boswell Exam
VWO Mathematics A – Practice exam 1
Solution key

Date:	
Time:	3 hours
Number of questions:	6
Number of subquestions:	24
Number of supplements:	0
Total score:	72

Subject-specific marking rules and guidelines

1. For each error or mistake in calculation a single point will be subtracted from the maximum score that can be obtained for that particular part of the question.
2. If a required explanation, deduction or calculation has been omitted or has been stated incorrectly 0 points will be awarded, unless otherwise stated in the solution key. This is also the case for answers obtained by the use of a graphic calculator. Answers obtained by the graphic calculator should indicate how the graphic calculator has been used to obtain the answer. Candidates must make sure they mention formulas applied or provide lists and calculation methods used in their answers.
3. If a notational error has been made, but the error can be seen to have no influence on the final result, no points will be deducted from the total score. If, however, it is not possible to determine that there is no influence on the final result a point will be deducted from the final score.
4. A particular mistake in the answer to a particular exam question will lead to a deduction of points only once, unless the question is substantially simplified by the mistake and/or when the solution key specifies otherwise.
5. A repeated mistake made in the answer to different exam questions will lead to a deduction of points each time such a mistake has been made, unless the solution key specifies otherwise.
6. If only one example, reason, explication, explanation or any other type of answer is required and more than one has been given, only the first answer given will be graded. If more than one example, reason, explication, explanation or any other type of answer is required, only the first answers are graded, up to and including the number of answers specified by the exam question.
7. If the candidate fails to give a required unit in the answer to a question a single point will be subtracted from the total score, unless the unit has been specified in the exam question.
8. If during intermediate steps results are rounded, resulting in an answer different from one in which non-rounded intermediate results are used, one point will be subtracted from the total score. Rounded intermediate results may, however, be noted down
Exceptions to this rule are those cases in which the context of the question requires the rounding of intermediate results. The maximum number of points deducted from the total score due to rounding errors is 2 for the entire exam.

Examples for the exceptions to rule 8.
Rounding off intermediate results can be forced by the context if, for example
- The amount of money for a single good has to be rounded to two decimals;
- The number of persons, things, etc. in a concrete situation (i.e. not for an average or expected value) has to be rounded to the nearest integer.

A required level of accuracy can be forced by the context if, for example
- The answer would not be distinguishable from a trivial answer. This can occur with the rounding of growth factors or probabilities to 0 or 1. A probability of $\left(\frac{1}{6}\right)^5$ may be rounded to 0.0001 but not to 0.000.

The forced rounding up or down of answers can occur, for example
- If the exam question specifies a minimum or maximum amount. (For example, if the question is: ‘What is the minimal distance an athlete has to jump to gain a certain number of points in a contest?’)

The above examples by no means exhaust all possible cases.

Question 1: Ice skating

a	$\binom{7}{3} = 35$ possibilities. If the candidate calculates the number of combinations of (3 or 4) out of 8 a maximum of one point may be awarded to this particular part of the question.	2
b	If you count Sven and Kjeld as a single contestant, there are $4! = 24$ different 'trains'.	1
	Sven and Kjeld can be arranged in $(2! =)$ 2 possible ways amongst each other	1
	So the answer is: $2 \cdot 24 = 48$ 'trains'. If the candidate uses wrong numbers during the calculation but applies the multiplication rule correctly, one point may be awarded to this particular part of the question.	1
c	$Y =$ number of competitions won by Rafa. Insight that: $Y \sim Bin(5, 0.6)$.	1
	$P(Y > 2) = 1 - binomcdf(5, 0.6, 2) \approx 0.683$.	1
d	$P(X = 3) = P(RRR) + P(PPP) = 0.6^3 + 0.4^3 = 0.28$.	1
	Calculating $P(X = 4)$. *Method 1: (using the complement rule)* $P(X = 4) = 1 - P(X = 3) - P(X = 5) = 1 - 0.28 - 0.3456 = 0.3744$. *Method* 2: *(direct calculation)* $P(X = 4) = 3 \cdot P(PRRR) + 3 \cdot P(RPPP) = 3 \cdot 0.4 \cdot 0.6^3 + 3 \cdot 0.6 \cdot 0.4^3 = 0.3744$.	1 1
	The probability distribution of X is: <table><tr><td>k</td><td>3</td><td>4</td><td>5</td></tr><tr><td>$P(X = k)$</td><td>0.28</td><td>0.3744</td><td>0.3456</td></tr></table>	
	$E(X) = 3 \cdot 0.28 + 4 \cdot 0.3744 + 5 \cdot 0.3456$	1
	The answer: 4.1 rounds.	1
e	Insight that the common ratio (growth factor) is equal to 1.007.	1
	$u_n = 30.4 \cdot 1.007^{n-1}$	1
f	*Method 1:*	
	Calculating the amount of seconds needed to complete 7 full rounds: $\left(\sum_{i=1}^{7} u_n = \frac{u_8 - u_1}{1.007 - 1} =\right) \frac{30.4 \cdot 1.007^7 - 30.4}{0.007} \approx 217.3$	2
	The total amount of time equals $217.3 + 20.6 = 237.9$ seconds. This means that she completes the training in less than 4 minutes (because $237.9 < 240$).	1
	Method 2:	
	Calculating the amount of seconds needed to complete 7 full rounds: $u_1 + u_2 + \cdots + u_7 = 30.4 + 30.6 \ldots + \ \ldots \ + 31.6 \ldots \approx 217.3$ (where all individual terms have been either calculated or determined using a table in the graphing calculator)	2
	The total time is therefore $217.3 + 20.6 = 237.9$ seconds. This means that she completes the training in less than 4 minutes (because $237.9 < 240$).	1

Question 2: Vocabulary

a	Method 1:	
	$g_{\text{year}} = \mathrm{e}^{0.239}$ $(= 1.269 \ldots)$	1
	$g_{\text{month}} = (\mathrm{e}^{0.239})^{\frac{1}{12}} \approx 1.020$	1
	So the vocabulary grows with 2.0% each month.	1
	Method 2:	
	After one month, the vocabulary equals $17\,000 \cdot \mathrm{e}^{0.239 \cdot \frac{1}{12}}$ $(\approx 17\,342)$	1
	$\frac{17\,342 - 17\,000}{170\,000} \cdot 100\% \approx 2.0\%$ (so the vocabulary grows with 2.0% each month).	2
b	Insight that the equation $17\,000 \cdot \mathrm{e}^{0.239t} = 42\,500$ has to be solved.	1
	Description of how this equation can be solved. • Algebraically: ○ $\mathrm{e}^{0.239t} = \frac{42\,500}{17\,000} = 2.5$ ○ $0.239t = \ln(2.5)$ ○ $t = \frac{1}{0.239} \cdot \ln(2.5) \approx 3.8$ • Graphic-numerically: ○ $Y_1 = 17\,000 \cdot \mathrm{e}^{0.239x}$ and $Y_2 = 42\,500$ ○ Option intersect yields $x \approx 3.8$ (or more accurately)	1 1
	42 500 words correspond to $(12 + 3 =)$ 15 years and $0.8 \cdot 12 \approx 10$ month.	1
	Humam is 22 months younger than expected	1
c	$a \left(= \frac{\Delta W}{\Delta t}\right) = \frac{45000 - 17000}{21 - 12} \approx 3111$	1
	b is the amount of vocabulary at the age of 12: $b = 17\,000$, so $W_S = 3111t + 17\,000$	1
d	Insight that it has to be the case that: $17\,000 \cdot \mathrm{e}^{0.239t} = 2(3111t + 17\,000)$	1
	Description of how this equation can be solved. • Graphic-numerically: ○ $Y_1 = 17\,000 \cdot \mathrm{e}^{0.239x}$ and $Y_2 = 2 \cdot (3111x + 17\,000)$ ○ Option intersect yields $x \approx 6$	1
	The answer: at the age of 18.	1

Question 3: Derivatives

a	Application of the quotient rule: $g'(x) = \dfrac{x \cdot 10 \cdot \frac{1}{x} - 10\ln(x) \cdot 1}{x^2} \left(= \dfrac{10 - 10\ln(x)}{x^2}\right)$	2
	(The line is given by $k: y = ax + b$) $a = g'(1) = \frac{10 - 10\ln(1)}{1^2} = 10.$	1
	Substituting the coordinates of $A(1,0)$ into $y = 10x + b$ gives: $0 = 10 \cdot 1 + b$ so $b = -10.$ So $k: y = 10x - 10$	1
b	Application of the product rule: $h'(x) = [\mathrm{e}^{-0.75x}]' \cdot x^3 + \mathrm{e}^{-0.75x} \cdot [x^3]'$	1
	Application of the chain rule: $[\mathrm{e}^{-0.75x}]' = -0.75\mathrm{e}^{-0.75x}$	1
	$h'(x) = -0.75\mathrm{e}^{-0.75x} \cdot x^3 + \mathrm{e}^{-0.75x} \cdot 3x^2 \left(= \mathrm{e}^{-0.75x}(3x^2 - 0.75x^3)\right)$	1
	Insight that the equation $h'(x) = 0$ has to be solved.	1
	A description of how the equation can be solved. • Algebraically: ○ $(\mathrm{e}^{-0.75x} = 0 \vee)\ 3x^2 - 0{,}75x^3 = 0$ ○ ($\mathrm{e}^{-0.75x} = 0$ has no solutions, so) $x^2 \cdot (3 - 0{,}75x) = 0$ ○ $x^2 = 0 \vee 3 - 0.75x = 0$ ○ $x = 0 \vee x = \frac{3}{0.75} = 4$ • Graphical-numerically: ○ $Y_1 = \mathrm{e}^{-0.75x}(3x^2 - 0.75x^3)$ (or written differently) ○ Option zero yields $x = 0$ and $x = 4$	1 1
	A plot of the graph of h shows that: (At $x = 0$ the graph of h has no maximum or minimum point) At $x = 4$ the graph of h has a maximum point.	1

Question 4: Cucumbers

a	$X =$ length of a cucumber (in cm) $X \sim Norm(36.2, 5.7)$	
	$P(X \geq 25) = normalcdf(25, 10^{99}, 36.2, 5.7) \approx 0.975.$	1
	So 97.5% if his cucumbers agree with European standards	1
b	$Y =$ weight of cucumbers (in grams) $Y \sim Norm(216.5, \sigma)$	
	Method 1:	
	Insight that it has to be the case that $\mathrm{P}(Y < 180) = 0.10$	1
	$Y_1 = normalcdf(-10^{99}, 180, 216.5, X)$ $Y_2 = 0.10$ Option intersect gives $X \approx 28.5$	1
	So $\sigma = 28.5$ (grams)	1
	Method 2:	
	Insight that it has to be the case that $\mathrm{P}(Y < 180) = 0.10$	1
	$z = InvNorm(0.10, 1, 0) = -1.281 \ldots$	1
	$\sigma = \frac{180 - 216.5}{-1.281\ldots} \approx 28.5$ (grams)	1
c	$p =$ the probability that cucumber is 'bent'. If the candidate does not specify the meaning of p one point will be subtracted from the score.	
	$H_0: p = 0.30$	1
	$H_a: p > 0.30$	1
d	$X =$ number of sold bent cucumbers during the experiment	
	Insight that if H_0 holds, $X \sim Bin(2958, 0.30)$	1
	Calculating the p-value: $\mathrm{P}(X \geq 1053) = 1 - P(X \leq 1052) = 1 - binomcdf(2958, 0.30, 1052)$ $\approx 3.48 \cdot 10^{-11}$	 1 1
	$3.48 \cdot 10^{-11} < 0.05$. (so H_0 is rejected and H_a has been shown) It has been shown (with $\alpha = 0.05$) that putting the sign above the bent cucumbers has a positive effect on their sales	1

Question 5: Mass and metabolism

a	*Method 1:*	
	When the weight becomes $\frac{650}{11{,}5} = 56.52\ldots$ times as high, the energy usage becomes $\frac{530}{25{,}6} = 20.70\ldots$ times as high.	1
	$56.52\ldots \neq 20.70\ldots$, so the relationship is not directly proportional.	1
	Method 2:	
	The energy usage to weight ratio is given by: $\frac{25{,}6}{11{,}5} = 2.226\ldots$ en $\frac{530}{650} = 0.185\ldots\ (\text{W/kg})$	1
	$2.226\ldots \neq 0.185\ldots$, so the relationship is not directly proportional.	1
	Method 3:	
	For a directly proportional relationship it needs to be the case that $E = c \cdot m$ From $m = 11.5$ and $E = 25.6$ it follows that $c = \frac{25.6}{11.5} = 2.226\ldots$ (so $E = 2.226\ldots \cdot m$)	1
	$m = 650$ implies $E = 2.226\ldots \cdot 650 \approx 1447 \neq 530$, so the relationship cannot be directly proportional.	1
b	Insight that the equation $25.6 = 4.1 \cdot 11.5^a$ has to be solved.	1

	A description of how this equation can be solved.
	• Algebraically:
	  ○ $11.5^a = \frac{25.6}{4.1}$
	  ○ $a = {}^{11.5}\log\left(\frac{25.6}{4.1}\right)$
	• Graphic-numerically:
	  ○ $Y_1 = 4.1 \cdot 11.5^x$ and $Y_2 = 11.5$
	  ○ Option intersect yields $x \approx 0.750$ (or more accurately).
		1
		
		1

	The answer: $a \approx 0.750$.	1
c	Insight that the equation $100 = 4.1 \cdot m^{0.75(0)}$ has to be solved.	1

	A description of how this equation can be solved.
	• Algebraically:
	  ○ $m^{0.75} = \frac{100}{4.1}$
	  ○ $m = \left(\frac{100}{4.1}\right)^{\frac{1}{0.75}}$
	• Graphic-numerically:
	  ○ $Y_1 = 4.1 \cdot x^{0.75}$ and $Y_2 = 100$
	  ○ Option intersect yields $x \approx 70.7$ (or more accurately).
		1
		
		1

	The answer: 71 (kg) (rounded to the nearest number)	1
d	$\log(E) = \log(4.1) + \log(m^{0.75})$ (application of ${}^g\log(a) + {}^g\log(b) = {}^g\log(ab)$)	1
	$\log(E) = \log(4.1) + 0.75 \cdot \log(m)$ (application of ${}^g\log(a^k) = k \cdot {}^g\log(a)$)	1
	$\log(E) \approx 0.61 + 0.75 \cdot \log(m)$	1
e	$E_r = \dfrac{4.1 \cdot m^{0.75}}{m}$	1
	$E_r = 4.1 \cdot \dfrac{m^{0.75}}{m} = 4.1 \cdot m^{0.75-1} = 4.1 \cdot m^{-0.25}$ If the candidate has only written $E_r = 4.1 \cdot m^{-0.25}$ no points should be awarded to this part of the answer.	1
f	$E_r'(m) = -0.25 \cdot 4.1 \cdot m^{-1.25} (= -1.025 \cdot m^{-1.25})$	1
	The value of $m^{-1.25}$ is always positive.	1
	Therefore the value of $-1.025 \cdot m^{-1.25}$ is always negative. $E_r'(m)$ is negative for all values of m (so the graph of E_r decreases for all values of m).	1

Question 6: Owls and mice

<table>
<tr><td>a</td><td>Method 1:</td><td></td></tr>
<tr><td></td><td>(The inequality $1000 + 200 \cdot \sin\left(\frac{1}{12}\pi t\right) > 1150$ has to be solved.)</td><td></td></tr>
<tr><td></td><td>A description of how the equation $1000 + 200 \cdot \sin\left(\frac{1}{12}\pi t\right) = 1150$ can be solved.</td><td>1</td></tr>
<tr><td></td><td>On the domain $[0,12]$ the solutions are $t \approx 3.24$ and $t \approx 8.76$</td><td>1</td></tr>
<tr><td></td><td>So from May up until and including August there are more than 1150 mice in the area during the entire month.</td><td>2</td></tr>
<tr><td></td><td>Method 2:</td><td></td></tr>
<tr><td></td><td>(The inequality $1000 + 200 \cdot \sin\left(\frac{1}{12}\pi t\right) > 1150$ has to be solved.)</td><td></td></tr>
<tr><td></td><td>$Y_1 = 1000 + 200 \cdot \sin\left(\frac{1}{12}\pi x\right)$
From the table follows:
<table>
<tr><th>t</th><th>$M(t)$</th></tr>
<tr><td>3</td><td>1141</td></tr>
<tr><td>4</td><td>1173</td></tr>
<tr><td>8</td><td>1173</td></tr>
<tr><td>9</td><td>1141</td></tr>
</table>
(Note: the candidate needs to show when the transitions occur.)</td><td>2</td></tr>
<tr><td></td><td>So from May up until and including August there are more than 1150 mice in the area during the entire month.</td><td>2</td></tr>
<tr><td>b</td><td>$a = \frac{400+200}{2} = 300$ (this is the average value (the height of the sinusoidal axis))
$b = 400 - 300 = 100$ (this is the amplitude)</td><td>1</td></tr>
<tr><td></td><td>The period is 24 months (equal tot the period of the graph of $M(t)$),
so $c = \frac{2\pi}{24}$ $(= \frac{1}{12}\pi \approx 0{,}26$ (or more accurate))</td><td>1</td></tr>
<tr><td></td><td>$d = 6$ (plus or minus a multiple of 24). This is the t-coordinate of a point where the graph intersects the sinusoidal axis under a positive slope.</td><td>1</td></tr>
</table>